Basic Concepts and Definitions

[1] The general form of P.D.E of 2^{nd} order in 2-variables:

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

or

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G$$

[2] types of 2^{nd} order P.D.E:

[1] Linear 2^{nd} order P.D.E:

- if A, B, C, D, E, F, G are functions in (x, y) or constants.

$$A(x,y) u_{xx} + B(x,y) u_{xy} + C(x,y) u_{yy} + D(x,y) u_x + E(x,y) u_y + F(x,y) u = G(x,y)$$

[2] semilinear 2^{nd} order P.D.E:

- if A, B, C, D, E are functions in (x, y) or constants
  F, G are non linear function in (u) i.e. $[u^2, u^3, \sin u, e^u, \dots]$

$$A(x,y) u_{xx} + B(x,y) u_{xy} + C(x,y) u_{yy} + D(x,y) u_x + E(x,y) u_y = G(x,y,u) \longrightarrow$$ شرط لاتكون u فقط

[3] Almost Linear 2^{nd} order P.D.E:

- if highest derivative $\longrightarrow$ linear and lower derivative $\longrightarrow$ Non Linear in (u_x) or (u_y)

$$\underbrace{\left(A(x,y) u_{xx} + B(x,y) u_{xy} + C(x,y) u_{yy}\right)}_{\text{highest derivative}} + \underbrace{\left(G(x,y,u,u_x,u_y)\right)}_{\text{lower derivative}} = 0$$

[4] Quasi Linear 2^{nd} order P.D.E:

- if

$$A(x,y,u,u_x,u_y) u_{xx} + B(x,y,u,u_x,u_y) u_{xy} + C(x,y,u,u_x,u_y) u_{yy} + G(x,y,u,u_x,u_y) = 0$$

[5] Non linear 2^{nd} order P.D.E:

- if it is neither linear nor semilinear nor almost nor Quasi Linear

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[3] Homogenous and Non homogenous:

- if $G = 0 \longrightarrow$ homogenous
- if $G \neq 0 \longrightarrow$ Non-homogenous

[4] order and degree:

order: the highest ordered partial derivative appearing in the equation.

Degree: the highest power of the highest ordered partial derivative appearing in the equation

Vector operators:

[1] Nabla (Delta) [Del]:

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

[2] Gradient [Grad]:

$$\text{Grad}(f) = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$ where f is scalar field

$$\nabla(g.f) = f(\nabla.g) + g.(\nabla f)$$

$$\nabla(\bar{u}.\bar{v}) = (\nabla.\bar{u}).\bar{v} + (\nabla.\bar{v}).\bar{u} + \bar{u}\times(\nabla\times\bar{v}) + \bar{v}\times(\nabla\times\bar{u})$$

[3] Divergence [Div]:

$$\text{Div}(\bar{v}) = \nabla.\bar{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

- where $\bar{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ is vector field

$$\nabla(f.\bar{v}) = f(\nabla.\bar{v}) + \bar{v}(\nabla.f)$$

- If $\nabla.\bar{v} = 0 \rightarrow$ [Solenoidal vector field] or [Incompressible vector field]

$$\nabla(\bar{u}.\bar{v}) = \bar{v}(\nabla\times\bar{u}) - \bar{u}(\nabla\times\bar{v})$$

[4] [Curl] Rotation:

$$\text{Curl}(\bar{v}) = \nabla\times\bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

- If $\nabla\times\bar{v} = 0 \rightarrow$ [Irrotational vector field]

$$\nabla\times(f.\bar{v}) = (\nabla.f)\times\bar{v} + f(\nabla\times\bar{v})$$

$$\nabla\times(\bar{u}.\bar{v}) = \bar{u}(\nabla.\bar{v}) - \bar{v}(\nabla.\bar{u}) + (\bar{v}.\nabla)\bar{u} - (\bar{u}.\nabla)\bar{v}$$

[5] Directional derivative:

the directional derivative of a scalar field $f(x,y,z)$ in the direction $\bar{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ is defined as:

$$\bar{a}.\text{Grad}(f) = (\bar{a}.\nabla)f = a_x\frac{\partial f}{\partial x} + a_y\frac{\partial f}{\partial y} + a_z\frac{\partial f}{\partial z}$$

this gives the rate of change of f in the direction of $\bar{a}$

[6] Laplacian:

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- If $\Delta\phi = 0 \rightarrow \therefore \phi$ is harmonic function

[7] Product rules:

$$\nabla(f.g) = f\,\nabla.g + g\,\nabla.f$$

$$\nabla(\bar{u}.\bar{v}) = \bar{u}\times(\nabla\times\bar{v}) + \bar{v}\times(\nabla\times\bar{u}) + (\bar{u}.\nabla)\bar{v} + (\bar{v}.\nabla)\bar{u}$$

$$\nabla(f.\bar{v}) = f(\nabla.\bar{v}) + \bar{v}.(\nabla f)$$

$$\nabla(\bar{u}\times\bar{v}) = \bar{v}.(\nabla\times\bar{u}) - \bar{u}(\nabla\times\bar{v})$$

$$\nabla\times(f.\bar{v}) = (\nabla f)\times\bar{v} + f(\nabla\times\bar{v})$$

$$\nabla\times(\bar{u}\times\bar{v}) = \bar{u}(\nabla.\bar{v}) - \bar{v}(\nabla.\bar{u}) + (\bar{v}.\nabla)\bar{u} - (\bar{u}.\nabla)\bar{v}$$

1. For each of the following, state whether the partial differential equation is linear, quasi-linear or non linear. If it is linear, state whether it is homogenous or non homogenous and gives its order

A. $u_{xx} + x u_y = y$
Linear - Non homogenous - order = 2 - Degree = 1

B. $u u_x - 2xy u_y = 0$
quasi-linear - order = 1 - Degree = 1

C. $u_x^2 + u u_y = 1$
Non-linear - order = 1 - Degree = 2

D. $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$
Linear - homogenous - order = 4 - Degree = 1

E. $u_{xx} + 2u_{xy} + u_{yy} = \sin x$
Linear - nonhomogenous - order = 2 - Degree = 1

F. $u_{xxx} + u_{xyy} + \log u = 0$
semi-linear - order = 3 - Degree = 1

G. $u_{xx}^2 + u_x^2 + \sin u = e^y$
Non-linear - order = 2 - Degree = 2

H. $u_t + u u_x + u_{xxx} = 0$
almost-linear - order = 3 - Degree = 1

2. verify that the functions $u(x,y) = x^2 - y^2$, $u(x,y) = e^x \sin y$, $u(x,y) = 2xy$ are the solutions of the equation $u_{xx} + u_{yy} = 0$

Solution

① $u(x,y) = x^2 - y^2$
$u_x = 2x$, $u_{xx} = 2$, $u_y = -2y$, $u_{yy} = -2$
$u_{xx} + u_{yy} = 2 - 2 = 0$

② $u(x,y) = e^x \sin y$
$u_x = e^x \sin y$, $u_{xx} = e^x \sin y$, $u_y = e^x \cos y$, $u_{yy} = -e^x \sin y$
$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$

③ $u(x,y) = 2xy$
$u_x = 2y$, $u_{xx} = 0$, $u_y = 2x$, $u_{yy} = 0$
$u_{xx} + u_{yy} = 0 + 0 = 0$

3. Show that $u = f(xy)$, where f is an arbitrary differentiable function satisfies $x u_x - y u_y = 0$ and verify that the functions $\sin(xy)$, $\cos(xy)$, $\ln(xy)$, e^{xy} are solutions.

Solution

① $u = f(z)$, $z = xy$

$u_x = u_z \cdot z_x$, $z_x = y \longrightarrow u_x = y \cdot u_z$

$u_y = u_z \cdot z_y$, $z_y = x \longrightarrow u_y = x \cdot u_z$

$x u_x - y u_y = xy\, u_z - xy\, u_z = 0$

② ⓐ if $u = \sin(xy)$

$u_x = y \cos(xy)$, $u_y = x \cos(xy)$

$x u_x - y u_y = xy \cos(xy) - xy \cos(xy) = 0$

ⓑ if $u = \cos(xy)$

$u_x = -y \sin(xy)$, $u_y = -x \sin(xy)$

$x u_x - y u_y = -xy \sin(xy) + xy \sin(xy) = 0$

ⓒ if $u = \ln(xy)$

$u_x = \frac{1}{x}$, $u_y = \frac{1}{y}$

$x u_x - y u_y = x \cdot \frac{1}{x} - y \cdot \frac{1}{y} = 0$

ⓓ if $u = e^{xy}$

$u_x = y e^{xy}$, $u_y = x e^{xy}$

$x u_x - y u_y = xy\, e^{xy} - xy\, e^{xy} = 0$

4. Show that $u = f(x) \cdot g(y)$ where f and g are arbitrary twice differentiable functions satisfies $u \cdot u_{xy} - u_x u_y = 0$

Solution

$u_x = g \cdot f_x$, $u_y = f \cdot g_y$, $u_{xy} = f_x \cdot g_y$

$u \cdot u_{xy} - u_x \cdot u_y = f \cdot g \cdot f_x \cdot g_y - g \cdot f_x \cdot f \cdot g_y = 0$